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Integrable nonlinear ladder system with background-controlled intersite resonant coupling

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Abstract

A new spectral problem on one-dimensional lattices is found allowing consistently to support the zero-curvature representation for a wide class of integrable nonlinear ladder systems. The modified recurrence technique for obtaining an infinite set of conservation laws is developed and some basic conserved quantities are explicitly derived. The eigenvalue problems associated with the limiting spectral operator for the special case of rapidly vanishing boundary conditions on Schrödinger-type fields and finite background condition on a concomitant field are solved and the domains of analyticity of Jost functions are presented both analytically and graphically. This particular example shows that the original auxiliary spectral problem is basically of fourth order and must generate a set of four distinct Jost functions that have to be involved in the procedure of inverse scattering transform. Moreover, there exists a critical background value of accompanying field which separates two principally different possibilities in the organization of analyticity domains of Jost functions. This crossover should inevitably lead to qualitative rearrangements in the structure of model solutions. Thus already in the limit of low-amplitude excitations we strictly observe the loss of stability regarding the linear spectrum of Schrödinger subsystem just above the critical background value of practically unexcited concomitant field, whereas in the stability region the structure of linear spectrum is essentially controlled by the magnitude of background level via effective modification of both intersite resonant coupling and self-site coupling.

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1. Introduction

When beginning work on this paper our main aim was to develop the integrable nonlinear system on a ladder lattice that should incorporate the fields akin to probability amplitudes, as in already known semidiscretized nonlinear Schrödinger systems [1–3], and include some concomitant fields of absolutely different origin. The development of second-order auxiliary spectral problems (or simply exploration of the known ones) to proceed with this goal was checked to be inappropriate. Hence we had to propose some auxiliary spectral problem of higher order distinguished, on the one hand, by its constructivity (the auxiliary spectral and evolution problems must produce the meaningful compatibility condition [4]) and, on the other hand, by its clear tangibility (the size of isolated nonlinear model must be as short as possible). The main results of these efforts are presented in this paper.

The extensive descriptions of driving forces regarding our present activity as well as the relevant references on the classics of solitonic theory are given in our previous articles [5–7] so we can safely omit them here.

2. Searching for the spectral $\mathcal{L}(n|z)$ and evolution $\mathcal{A}(n|z)$ operators

One of the recognized ways for developing the integrable nonlinear dynamical models on one-dimensional [1–4] and quasi-one-dimensional [5, 6] lattices consists of an appropriate choice of spectral operator $\mathcal{L}(n|z)$ in order for the zero-curvature equation

$$\dot{\mathcal{L}}(n|z) = \mathcal{A}(n+1|z)\mathcal{L}(n|z) - \mathcal{L}(n|z)\mathcal{A}(n|z) \tag{1}$$

to become a meaningful one, i.e., the evolution operator $\mathcal{A}(n|z)$ to be determinable in terms of $\mathcal{L}(n|z)$ [4]. Here the integer n denotes the discrete coordinate variable running for the sake of definiteness from minus to plus infinity, the overdot stands for the differentiation with respect to time variable τ, while z marks the time-independent spectral parameter.

The spectral operator $\mathcal{L}(n|z)$ giving rise to the present paper has been derived by the method of successive unsuccessful attempts. Its original version is given by the following 4×4 matrix:

$$\mathcal{L}(n|z) = \begin{pmatrix} K_{11}(n)z^2 & K_{12}(n) & F_{13}(n)z & F_{14}(n)z^{-1} \\ K_{21}(n) & K_{22}(n)z^2 & F_{23}(n)z^{-1} & F_{24}(n)z \\ G_{31}(n)z^{-1} & G_{32}(n)z & K_{33}(n)z^{-2} & K_{34}(n) \\ G_{41}(n)z & G_{42}(n)z^{-1} & K_{43}(n) & K_{44}(n)z^{-2} \end{pmatrix} \tag{2}$$

where all entries after dropping the respective spectral multipliers are presumed to be distinct functions of coordinate n and time τ, thus playing the role of presupposed field variables. Although this form of spectral operator $\mathcal{L}(n|z)$ permits a self-consistent reconstruction of evolution operator $\mathcal{A}(n|z)$ in the framework of zero-curvature equation (1), the recovered evolution equations on field variables turn out to be immensely cumbersome. Nevertheless, the analysis of aforementioned evolution equations allows one to observe that the number of independent fields can be halved by an appropriate symmetrizing reduction lifting simultaneously some undesirable restrictions imposed by the desirable locality of theory.

Thus, there is every reason to dwell only upon the symmetrized version of spectral operator which we write in the block-matrix form

$$\mathcal{L}(n|z) = \begin{pmatrix} \mathrm{L}_{11}(n|z) & \mathrm{L}_{12}(n|z) \\ \mathrm{L}_{21}(n|z) & \mathrm{L}_{22}(n|z) \end{pmatrix} \tag{3}$$

with the 2×2 submatrices $\mathrm{L}_{jk}(n|z)$ specified by

$$\mathrm{L}_{jk}(n|z) = l^{+}_{jk}(n|z)\mathrm{I} + l^{-}_{jk}(n|z)\mathrm{T}. \tag{4}$$

Here I is the unity 2×2 matrix, T is the 2×2 matrix of property $\mathrm{T}^2 = \mathrm{I}$ (for example any of the Pauli matrices) and the shorthands

$$l_{11}^{+}(n|z) = K_{11}^{+}(n)z^2 \qquad l_{11}^{-}(n|z) = K_{11}^{-}(n) \tag{5}$$

$$l_{12}^{+}(n|z) = F_{12}^{+}(n)z \qquad l_{12}^{-}(n|z) = F_{12}^{-}(n)z^{-1} \tag{6}$$

$$l_{21}^{+}(n|z) = G_{21}^{+}(n)z^{-1} \qquad l_{21}^{-}(n|z) = G_{21}^{-}(n)z \tag{7}$$

$$l_{22}^{+}(n|z) = K_{22}^{+}(n)z^{-2} \qquad l_{22}^{-}(n|z) = K_{22}^{-}(n) \tag{8}$$

are admitted.

The similar block-matrix structure

$$\mathcal{A}(n|z) = \begin{pmatrix} \mathrm{A}_{11}(n|z) & \mathrm{A}_{21}(n|z) \\ \mathrm{A}_{21}(n|z) & \mathrm{A}_{22}(n|z) \end{pmatrix} \tag{9}$$

with

$$\mathrm{A}_{jk}(n|z) = a_{jk}^{+}(n|z)\mathrm{I} + a_{jk}^{-}(n|z)\mathrm{T} \tag{10}$$

can be adopted for the evolution operator when seeking its matrix elements. In order to explicitly obtain the lowest feasible evolution operator in an infinite hierarchy we will follow the well-approbated mnemonic rule [6, 8] and assume that each matrix element of $\mathcal{A}(n|z)$ must carry the same power of spectral parameter z as its counterpart from the squared spectral operator $\mathcal{L}^2(n|z)$.

One can easily verify that this demand is tantamount to the ansätze

$$a_{11}^{+}(n|z) = a_{11}^{+}(n)z^4 + b_{11}^{+}(n) \qquad a_{11}^{-}(n|z) = a_{11}^{-}(n)z^2 + b_{11}^{-}(n)z^{-2} \tag{11}$$

$$a_{12}^{+}(n|z) = a_{12}^{+}(n)z^3 + b_{12}^{+}(n)z^{-1} \qquad a_{12}^{-}(n|z) = a_{12}^{-}(n)z + b_{12}^{-}(n)z^{-3} \tag{12}$$

$$a_{21}^{+}(n|z) = a_{21}^{+}(n)z + b_{21}^{+}(n)z^{-3} \qquad a_{21}^{-}(n|z) = a_{21}^{-}(n)z^3 + b_{21}^{-}(n)z^{-1} \tag{13}$$

$$a_{22}^{+}(n|z) = a_{22}^{+}(n) + b_{22}^{+}(n)z^{-4} \qquad a_{22}^{-}(n|z) = a_{22}^{-}(n)z^2 + b_{22}^{-}(n)z^{-2} \tag{14}$$

where the functions (of coordinate and time) $a_{jk}^{\pm}(n)$ and $b_{jk}^{\pm}(n)$ are as yet unknown. The straightforward but slightly tedious calculations based on the zero-curvature relation (1) with the use of representations (5)–(8) and (11)–(14) for the entries of spectral (3) and evolution (9) operators yield

$$a_{11}^{+}(n) = a_{11}^{+} \tag{15}$$

$$a_{12}^{+}(n) = a_{11}^{+} f_{12}^{+}(n) \tag{16}$$

$$a_{21}^{-}(n) = g_{21}^{-}(n-1)a_{11}^{+} \tag{17}$$

$$a_{11}^{-}(n) = a_{11}^{-} - a_{11}^{+} f_{12}^{+}(n) g_{21}^{-}(n-1) \tag{18}$$

$$a_{22}^{-}(n) = a_{22}^{-} + g_{21}^{-}(n-1)a_{11}^{+} f_{12}^{+}(n) \tag{19}$$

$$\begin{aligned} a_{12}^{-}(n) = {} & a_{11}^{+} f_{12}^{-}(n)\varkappa_{22}^{+}(n) + \left[a_{11}^{-} - a_{22}^{-} - a_{11}^{+}\varkappa_{11}^{-}(n)\right] f_{12}^{+}(n) + a_{11}^{+} f_{12}^{+}(n+1)\varkappa_{22}^{-}(n)\varkappa_{22}^{+}(n) \\ & - a_{11}^{+}\left[f_{12}^{+}(n+1)g_{21}^{-}(n) + f_{12}^{+}(n)g_{21}^{-}(n-1)\right] f_{12}^{+}(n) \end{aligned} \tag{20}$$

$$\begin{aligned} a_{21}^{+}(n) = {} & \varkappa_{22}^{+}(n-1)g_{21}^{+}(n-1)a_{11}^{+} + g_{21}^{-}(n-1)\left[a_{11}^{-} - a_{22}^{-} - \varkappa_{11}^{-}(n-1)a_{11}^{+}\right] \\ & + \varkappa_{22}^{+}(n-1)\varkappa_{22}^{-}(n-1)g_{21}^{-}(n-2)a_{11}^{+} \\ & - g_{21}^{-}(n-1)\left[f_{12}^{+}(n)g_{21}^{-}(n-1) + f_{12}^{+}(n-1)g_{21}^{-}(n-2)\right]a_{11}^{+} \end{aligned} \tag{21}$$

$$b^+_{22}(n) = b^+_{22} \tag{22}$$

$$b^+_{21}(n) = b^+_{22} g^+_{21}(n) \tag{23}$$

$$b^-_{12}(n) = f^-_{12}(n-1) b^+_{22} \tag{24}$$

$$b^-_{22}(n) = b^-_{22} - b^+_{22} g^+_{21}(n) f^-_{12}(n-1) \tag{25}$$

$$b^-_{11}(n) = b^-_{11} + f^-_{12}(n-1) b^+_{22} g^+_{21}(n) \tag{26}$$

$$\begin{aligned} b^-_{21}(n) = {} & b^+_{22} g^-_{21}(n) \varkappa^+_{11}(n) + \left[b^-_{22} - b^-_{11} - b^+_{22}\varkappa^-_{22}(n)\right] g^+_{21}(n) + b^+_{22} g^+_{21}(n+1) \varkappa^-_{11}(n) \varkappa^+_{11}(n) \\ & - b^+_{22}\left[g^+_{21}(n+1) f^-_{12}(n) + g^+_{21}(n) f^-_{12}(n-1)\right] g^+_{21}(n) \end{aligned} \tag{27}$$

$$\begin{aligned} b^+_{12}(n) = {} & \varkappa^+_{11}(n-1) f^+_{12}(n-1) b^+_{22} + f^-_{12}(n-1)\left[b^-_{22} - b^-_{11} - \varkappa^-_{22}(n-1) b^+_{22}\right] \\ & + \varkappa^+_{11}(n-1) \varkappa^-_{11}(n-1) f^-_{12}(n-2) b^+_{22} \\ & - f^-_{12}(n-1)\left[g^+_{21}(n) f^-_{12}(n-1) + g^+_{21}(n-1) g^-_{12}(n-2)\right] b^+_{22} \end{aligned} \tag{28}$$

where the quantities $a^+_{11}, a^-_{11}, a^-_{22}, b^+_{22}, b^-_{22}, b^-_{11}$ can, in principle, be arbitrary functions of time τ and the notations

$$f^+_{12}(n) = \frac{F^+_{12}(n)}{K^+_{11}(n)} \qquad g^+_{21}(n) = \frac{G^+_{21}(n)}{K^+_{22}(n)} \tag{29}$$

$$f^-_{12}(n) = \frac{F^-_{12}(n)}{K^+_{22}(n)} \qquad g^-_{21}(n) = \frac{G^-_{21}(n)}{K^+_{11}(n)} \tag{30}$$

$$\varkappa^-_{11}(n) = \frac{K^-_{11}(n)}{K^+_{11}(n)} \qquad \varkappa^-_{22}(n) = \frac{K^-_{22}(n)}{K^+_{22}(n)} \tag{31}$$

$$\varkappa^+_{11}(n) = \frac{K^+_{11}(n)}{K^+_{22}(n)} \qquad \varkappa^+_{22}(n) = \frac{K^+_{22}(n)}{K^+_{11}(n)} \tag{32}$$

are implied. It is important to emphasize that at this stage the functions $a^+_{22}(n)$ and $b^+_{11}(n)$ remain unspecified similarly to the situation with the generalized Ablowitz–Ladik system proposed by Tsuchida [9].

As for the differential-difference nonlinear equations on field variables that follow from the zero-curvature relation (1) we prefer to present them here only in the most general form

$$\begin{aligned} \dot{K}^+_{11}(n) = {} & b^+_{11}(n+1) K^+_{11}(n) + a^-_{11}(n+1) K^-_{11}(n) + a^+_{12}(n+1) G^+_{21}(n) + a^-_{12}(n+1) G^-_{21}(n) \\ & - K^+_{11}(n) b^+_{11}(n) - K^-_{11}(n) a^-_{11}(n) - F^+_{12}(n) a^+_{21}(n) - F^-_{12}(n) a^-_{21}(n) \end{aligned} \tag{33}$$

$$\begin{aligned} \dot{K}^-_{11}(n) = {} & b^+_{11}(n+1) K^-_{11}(n) + b^-_{11}(n+1) K^+_{11}(n) + b^+_{12}(n+1) G^-_{21}(n) + a^-_{12}(n+1) G^+_{21}(n) \\ & - K^+_{11}(n) b^-_{11}(n) - K^-_{11}(n) b^+_{11}(n) - F^+_{12}(n) b^-_{21}(n) - F^-_{12}(n) a^+_{21}(n) \end{aligned} \tag{34}$$

$$\begin{aligned} \dot{F}^+_{12}(n) = {} & b^+_{11}(n+1) F^+_{12}(n) + a^-_{11}(n+1) F^-_{12}(n) + a^+_{12}(n+1) K^+_{22}(n) + a^-_{12}(n+1) K^-_{22}(n) \\ & - K^+_{11}(n) b^+_{12}(n) - K^-_{11}(n) a^-_{12}(n) - F^+_{12}(n) a^+_{22}(n) - F^-_{12}(n) a^-_{22}(n) \end{aligned} \tag{35}$$

$$\begin{aligned} \dot{F}^-_{12}(n) = {} & b^+_{11}(n+1) F^-_{12}(n) + b^-_{11}(n+1) F^+_{12}(n) + b^+_{12}(n+1) K^-_{22}(n) + a^-_{12}(n+1) K^+_{22}(n) \\ & - K^+_{11}(n) b^-_{12}(n) - K^-_{11}(n) b^+_{12}(n) - F^+_{12}(n) b^-_{22}(n) - F^-_{12}(n) a^+_{22}(n) \end{aligned} \tag{36}$$

$$\begin{aligned} \dot{G}^+_{21}(n) = {} & b^+_{21}(n+1) K^+_{11}(n) + b^-_{21}(n+1) K^-_{11}(n) + a^+_{22}(n+1) G^+_{21}(n) + b^-_{22}(n+1) G^-_{21}(n) \\ & - G^+_{21}(n) b^+_{11}(n) - G^-_{21}(n) b^-_{11}(n) - K^+_{22}(n) a^+_{21}(n) - K^-_{22}(n) b^-_{21}(n) \end{aligned} \tag{37}$$

$$\begin{aligned} \dot{G}^-_{21}(n) = {} & a^+_{21}(n+1) K^-_{11}(n) + b^-_{21}(n+1) K^+_{11}(n) + a^+_{22}(n+1) G^-_{21}(n) + a^-_{22}(n+1) G^+_{21}(n) \\ & - G^+_{21}(n) a^-_{11}(n) - G^-_{21}(n) b^+_{11}(n) - K^+_{22}(n) a^-_{21}(n) - K^-_{22}(n) a^+_{21}(n) \end{aligned} \tag{38}$$

$$\dot{K}^{+}_{22}(n) = b^{+}_{21}(n+1)F^{+}_{12}(n) + b^{-}_{21}(n+1)F^{-}_{12}(n) + a^{+}_{22}(n+1)K^{+}_{22}(n) + b^{-}_{22}(n+1)K^{-}_{22}(n) - G^{+}_{21}(n)b^{+}_{12}(n) - G^{-}_{21}(n)b^{-}_{12}(n) - K^{+}_{22}(n)a^{+}_{22}(n) - K^{-}_{22}(n)b^{-}_{22}(n) \tag{39}$$

$$\dot{K}^{-}_{22}(n) = a^{+}_{21}(n+1)F^{-}_{12}(n) + b^{-}_{21}(n+1)F^{+}_{12}(n) + a^{+}_{22}(n+1)K^{-}_{22}(n) + a^{-}_{22}(n+1)K^{+}_{22}(n) - G^{+}_{21}(n)a^{-}_{12}(n) - G^{-}_{21}(n)b^{+}_{12}(n) - K^{+}_{22}(n)a^{-}_{22}(n) - K^{-}_{22}(n)a^{+}_{22}(n) \tag{40}$$

i.e., written without an explicit substitution of just obtained expressions (15)–(28) for the 14 specifiable functions amongst 16 incorporated ones $a^{\pm}_{jk}(n)$ and $b^{\pm}_{jk}(n)$.

3. How to obtain the conservation laws

Let us now try to obtain the conservation laws of our model (33)–(40) generalizing the Tsuchida–Ujino–Wadati approach [10, 11] in such a way as to rely upon the symmetries (4) and (10) of adopted spectral (3) and evolution (9) operators $\mathcal{L}(n|z)$ and $\mathcal{A}(n|z)$. For this purpose, we invoke the block-matrix version of auxiliary linear problems

$$|u(n+1|z)\rangle\rangle = \mathcal{L}(n|z)|u(n|z)\rangle\rangle \tag{41}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}|u(n|z)\rangle\rangle = \mathcal{A}(n|z)|u(n|z)\rangle\rangle \tag{42}$$

recognized as a version where the standard four-component one-column matrix (column vector) $|u(n|z)\rangle$ has been replaced by the two-column one $|u(n|z)\rangle\rangle$. In other words,

$$|u(n|z)\rangle\rangle = \begin{pmatrix} \mathrm{U}_1(n|z) \\ \mathrm{U}_2(n|z) \end{pmatrix} \tag{43}$$

where $\mathrm{U}_1(n|z)$ and $\mathrm{U}_2(n|z)$ are assumed to be time-dependent 2×2 submatrices.

Inasmuch as the accepted symmetry of 2×2 submatrices $\mathrm{L}_{jk}(n|z)$ and $\mathrm{A}_{jk}(n|z)$ implies the restricted two-matrix basis (i.e., basis consisting of two 2×2 matrices I and T) to be sufficient for their expansions (4) and (10) the similar I&T representation can also be justified for any other 2×2 submatrix appearing in our consideration. As a result, the algebra of permissible 2×2 submatrices associated with the symmetrized version of our model proves to be commutative one.

Now it is a ripe time to introduce some 2×2 matrix-valued quantities we have to rely upon when extracting information encoded in auxiliary linear problems (41) and (42) and in their compatibility (zero-curvature) condition (1). They are as follows:

$$\Gamma_{jk}(n|z) = \mathrm{U}_j(n|z)\mathrm{V}_k(n|z) \tag{44}$$

$$\mathrm{M}_{jk}(n|z) = \sum_{i=1}^{2}\mathrm{L}_{ji}(n|z)\Gamma_{ik}(n|z) \tag{45}$$

$$\mathrm{B}_{jk}(n|z) = \sum_{i=1}^{2}\mathrm{A}_{ji}(n|z)\Gamma_{ik}(n|z) \tag{46}$$

where $\mathrm{V}_k(n|z)$ is the 2×2 matrix inverse to $\mathrm{U}_k(n|z)$, i.e. $\mathrm{U}_k(n|z)\mathrm{V}_k(n|z) = \mathrm{I}$. Then passing through the equations

$$\dot{\Gamma}_{jk}(n|z) = \mathrm{B}_{jk}(n|z) - \Gamma_{jk}(n|z)\mathrm{B}_{kk}(n|z) \tag{47}$$

$$\Gamma_{jl}(n+1|z)\mathrm{M}_{lk}(n|z) = \mathrm{M}_{jk}(n|z) \tag{48}$$

we step by step come to

$$\dot{\mathrm{M}}_{jk}(n|z) = \mathrm{B}_{jl}(n+1|z)\mathrm{M}_{lk}(n|z) - \mathrm{M}_{jk}(n|z)\mathrm{B}_{kk}(n|z) \tag{49}$$

$$\mathrm{M}_{jl}(n+1|z)\mathrm{M}_{lk}(n|z) = \sum_{i=1}^{2} \mathrm{L}_{ji}(n+1|z)\mathrm{M}_{ik}(n|z). \tag{50}$$

Here the summation with respect to doubled index l is absent, whereas the mere summation, whenever it appears, is always marked by the standard summation symbol $\sum$.

The last two equations (49) and (50) accompanied by an appropriate trick with summation over the space coordinate n allow one to obtain the following two independent sets of equations:

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \sum_{n=-\infty}^{\infty} [\ln s_{jk}(n|z) + \ln s_{kj}(n|z)] = 0 \tag{51}$$

$$s_{jl}(n+1|z)s_{lk}(n|z) = \sum_{i=1}^{2} \sigma_{ji}(n+1|z)s_{ik}(n|z) \tag{52}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \sum_{n=-\infty}^{\infty} [\ln d_{jk}(n|z) + \ln d_{kj}(n|z)] = 0 \tag{53}$$

$$d_{jl}(n+1|z)d_{lk}(n|z) = \sum_{i=1}^{2} \delta_{ji}(n+1|z)d_{ik}(n|z) \tag{54}$$

where the matrices $\mathrm{B}_{jk}(n|z)$ dependent on evolution submatrices $\mathrm{A}_{jk}(n|z)$ are seen to be totally eliminated. Here the shorthand expressions

$$s_{jk}(n|z) = m^{+}_{jk}(n|z) + m^{-}_{jk}(n|z) \tag{55}$$

$$d_{jk}(n|z) = m^{+}_{jk}(n|z) - m^{-}_{jk}(n|z) \tag{56}$$

and

$$\sigma_{jk}(n|z) = l^{+}_{jk}(n|z) + l^{-}_{jk}(n|z) \tag{57}$$

$$\delta_{jk}(n|z) = l^{+}_{jk}(n|z) - l^{-}_{jk}(n|z) \tag{58}$$

with the functions $m^{+}_{jk}(n|z)$ and $m^{-}_{jk}(n|z)$ given by the expansion

$$\mathrm{M}_{jk}(n|z) = m^{+}_{jk}(n|z)\mathrm{I} + m^{-}_{jk}(n|z)\mathrm{T} \tag{59}$$

are tacitly adopted.

Each of the above-derived sets of basic equations (51) and (52) or (53) and (54), though being algebraically independent, gives rise to the same results when concerned with the model conservation laws. Hence, it is sufficient to analyse only first of them.

Thus, considering the second block of equations (52) from the first set we clearly observe the property

$$s_{12}(n|z)s_{21}(n|z) = s_{11}(n|z)s_{22}(n|z) \tag{60}$$

that permits to isolate another property

$$[s_{11}(n|z) - \sigma_{11}(n|z)]\,[s_{22}(n|z) - \sigma_{22}(n|z)] = \sigma_{12}(n|z)\sigma_{21}(n|z) \tag{61}$$

and as a result to initiate the very important substitutions

$$s_{11}(n|z) - \sigma_{11}(n|z) = \sigma_{12}(n|z)X_{21}(n|z) \tag{62}$$

$$s_{22}(n|z) - \sigma_{22}(n|z) = \sigma_{21}(n|z)X_{12}(n|z) \tag{63}$$

with the constraint

$$X_{12}(n|z)X_{21}(n|z) = 1 \tag{64}$$

on functions $X_{12}(n|z)$ and $X_{21}(n|z)$ being imposed.

The key repercussion of these artful efforts is to replace the original block (52) of rather ambiguous equations (s.s. due to their degeneration with respect to index l) by its essentially clearer counterpart

$$X_{21}(n+1|z)\sigma_{12}(n|z)X_{21}(n|z) + X_{21}(n+1|z)\sigma_{11}(n|z) - \sigma_{22}(n|z)X_{21}(n|z) = \sigma_{21}(n|z) \tag{65}$$

$$X_{12}(n+1|z)\sigma_{21}(n|z)X_{12}(n|z) + X_{12}(n+1|z)\sigma_{22}(n|z) - \sigma_{11}(n|z)X_{12}(n|z) = \sigma_{12}(n|z) \tag{66}$$

in which every equation contains only one unknown function. In view of invertibility condition (64), these equations are seen to be equivalent and the particular choice of either of them is dictated by the convenience reasons depending on the type of solution that we intend to fix via an admissible expansion in the vicinity of a distinguished point in the complex z-plane.

When dealing with the expansions near infinity or near the initial point there are indeed only four independent possibilities:

$$X_{21}(n|z) = z^{-1}\sum_{j=0}^{\infty} X_{21}(n|j|\infty)z^{-2j} \qquad (|z| \to \infty) \tag{67}$$

$$X_{12}(n|z) = z\sum_{j=0}^{\infty} X_{12}(n|j|0)z^{2j} \qquad (|z| \to 0) \tag{68}$$

$$X_{21}(n|z) = z\sum_{j=0}^{\infty} X_{21}(n|j|0)z^{2j} \qquad (|z| \to 0) \tag{69}$$

$$X_{12}(n|z) = z^{-1}\sum_{j=0}^{\infty} X_{12}(n|j|\infty)z^{-2j} \qquad (|z| \to \infty). \tag{70}$$

These expansions we call the basic expansions in contrast to the complementary ones emerging through the reversibility condition (64) and playing no new part in our consideration. Each of basic expansions (67)–(70), when inserted into its native equation (i.e., equation properly chosen among (65) and (66)), provides the foundation for developing the respective recurrence procedure and for recovering step by step more and more highly involved functional entries $X_{21}(n|j|\infty)$, $X_{12}(n|j|0)$, $X_{21}(n|j|0)$, $X_{12}(n|j|\infty)$. We will omit any detailed calculations in the framework of the recurrence approach saying only that their realization must invoke the explicit expressions for $\sigma_{jk}(n|z)$ given by (57) and (5)–(8).

Once the particular expansion for $X_{jk}(n|z)$ has been adopted and its several lowest terms were already found, the next steps are straightforward. (i) We must recover $s_{11}(n|z)$ if $X_{jk}(n|z) = X_{21}(n|z)$ or $s_{22}(n|z)$ if $X_{jk}(n|z) = X_{12}(n|z)$ with a sufficient accuracy relying on formula (62) or formula (63), respectively, accompanied by the explicit expressions for $\sigma_{jk}(n|z)$ as previously recommended. (ii) The result should be substituted into the generating equation (51) indexed suitably. (iii) Subsequent curtailed expansion of rightly indexed generating equation (51) produces several lowest conservation laws from the respective infinite set. Such a procedure ought to be repeated four times in accordance with the number of basic independent expansions (67)–(70).

Below we present four pairs of the lowest conservation laws yielded by the generating functions $\ln s_{11}(n|z)$ and $\ln s_{22}(n|z)$ through the respective generating equations (51)

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\sum_{n=-\infty}^{\infty}\ln K_{11}^{+}(n)=0 \tag{71}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\sum_{n=-\infty}^{\infty}\left[\varkappa_{11}^{-}(n)+f_{12}^{+}(n)g_{21}^{-}(n-1)\right]=0 \tag{72}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\sum_{n=-\infty}^{\infty}\ln K_{22}^{+}(n)=0 \tag{73}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\sum_{n=-\infty}^{\infty}\left[\varkappa_{22}^{-}(n)+g_{21}^{+}(n)f_{12}^{-}(n-1)\right]=0 \tag{74}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\sum_{n=-\infty}^{\infty}\ln\left[\varkappa_{11}^{-}(n)-f_{12}^{-}(n)\varkappa_{22}^{+}(n)g_{21}^{+}(n)\right]=0 \tag{75}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\sum_{n=-\infty}^{\infty}\frac{1+\varkappa_{11}^{-}(n)\varkappa_{22}^{-}(n)-f_{12}^{-}(n)g_{21}^{-}(n)-f_{12}^{+}(n)g_{21}^{+}(n)}{\varkappa_{11}^{-}(n)-f_{12}^{-}(n)\varkappa_{22}^{+}(n)g_{21}^{+}(n)}=0 \tag{76}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\sum_{n=-\infty}^{\infty}\ln\left[\varkappa_{22}^{-}(n)-g_{21}^{-}(n)\varkappa_{11}^{+}(n)f_{12}^{+}(n)\right]=0 \tag{77}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\sum_{n=-\infty}^{\infty}\frac{1+\varkappa_{22}^{-}(n)\varkappa_{11}^{-}(n)-g_{21}^{-}(n)f_{12}^{-}(n)-g_{21}^{+}(n)f_{12}^{+}(n)}{\varkappa_{22}^{-}(n)-g_{21}^{-}(n)\varkappa_{11}^{+}(n)f_{12}^{+}(n)}=0 \tag{78}$$

where the notations (29)–(32) are understood.

The use of generating function $\ln s_{12}(n|z)+\ln s_{21}(n|z)$ does not lead to any new independent conservation laws in view of its additive relationship with already considered generating functions $\ln s_{11}(n|z)$ and $\ln s_{22}(n|z)$ maintained by the earlier proved constriction (60).

4. Eigenvalue problems for the limiting spectral operator

The issue about the applicability of auxiliary spectral and evolution linear problems

$$|u(n+1|z)\rangle=\mathcal{L}(n|z)|u(n|z)\rangle \tag{79}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}|u(n|z)\rangle=\mathcal{A}(n|z)|u(n|z)\rangle \tag{80}$$

to the integration of associated nonlinear problem (33)–(40) remains to be abstract until the asymptotics of involved field functions at spatial infinities are left unsettled. This issue must inevitably be supplemented by the question how the as yet arbitrary gauge functions $a_{22}^{+}(n)$ and $b_{11}^{+}(n)$ have to be specified.

Below we will make only the first but basic step in the development of inverse scattering machinery and concentrate on some aspects of eigenvalue problems for the limiting spectral operator. As an example we will consider the reduction

$$K^{+}_{11}(n) = 1 \qquad K^{+}_{22}(n) = 1 \tag{81}$$

$$f^{+}_{12}(n) = +\mathrm{i}q_{+}(n) \qquad g^{+}_{21}(n) = +\mathrm{i}r_{+}(n) \tag{82}$$

$$f^{-}_{12}(n) = -\mathrm{i}q_{-}(n) \qquad g^{-}_{21}(n) = -\mathrm{i}r_{-}(n) \tag{83}$$

$$\varkappa^{-}_{11}(n) = \mu(n) \qquad \varkappa^{-}_{22}(n) = \nu(n) \tag{84}$$

with $q_{+}(n), r_{+}(n)$ and $q_{-}(n), r_{-}(n)$ rapidly vanishing at $|n| \to 0$ and

$$\lim_{|n|\to\infty} \mu(n) = \mu \qquad \lim_{|n|\to\infty} \nu(n) = \nu. \tag{85}$$

In this case, the limiting spectral operator

$$\mathcal{L}(z) = \lim_{|n|\to\infty} \mathcal{L}(n|z) \tag{86}$$

acquires the block-diagonal form and we can readily investigate the left

$$\mathcal{L}(z)|u(z)\rangle = |u(z)\rangle\zeta(z) \tag{87}$$

and the right

$$\langle u^{+}(z)|\mathcal{L}(z) = \zeta(z)\langle u^{+}(z)| \tag{88}$$

eigenvalue problems. The solution of these problems is known to be of key importance in Caudrey formulation of inverse scattering transform [7, 12, 13]. In particular, already the sole knowledge of eigenvalues

$$\zeta_1(z) = z^2 - \mu \tag{89}$$

$$\zeta_2(z) = z^2 + \mu \tag{90}$$

$$\zeta_3(z) = z^{-2} - \nu \tag{91}$$

$$\zeta_4(z) = z^{-2} + \nu \tag{92}$$

provides a very simple recipe how to divide the plane of complex spectral parameter z into the domains serving in future as the domains of analyticity of Jost functions. It (recipe) is based upon the fact that all four obtained eigenvalues (89)–(92) are distinct so that the eigenvalue problems (87) and (88) as well as the respective auxiliary spectral problem (79) should be treated as the fourth order ones. In this case, the lines between domains (boundary lines) are determined by the collection of six equations

$$|\zeta_j(z)| = |\zeta_k(z)| \tag{93}$$

where indices j and k span the integers from 1 to 4 in such a way as to prevent their mutual coincidence.

To further simplify the analysis we assume that the amplitudes $\mu(n)$ and $\nu(n)$ are complex-conjugated $\mu^{*}(n) = \nu(n)$. Such an additional reduction in the framework of our modelling is proved to be admissible what we will explicitly confirm in the next section. Then introducing the parametrization

$$\mu = \exp(\eta + \mathrm{i}\gamma) \tag{94}$$

$$\nu = \exp(\eta - \mathrm{i}\gamma) \tag{95}$$

$$z = \exp(\chi + \mathrm{i}\varphi + \mathrm{i}\gamma/2) \tag{96}$$

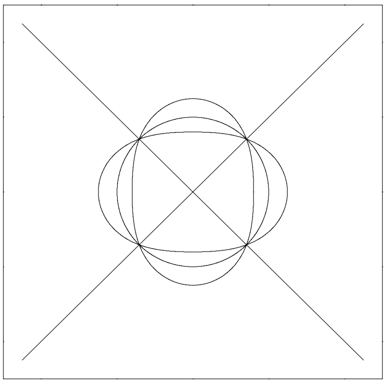

Figure 1. The plane of phase-adjusted spectral parameter $z\exp(-\mathrm{i}\gamma/2)$ and its typical subdivision into domains of analyticity of Jost functions at subcritical background levels of concomitant field $\mu\nu < 1$ (s.s. $\eta = -\pi/4$).

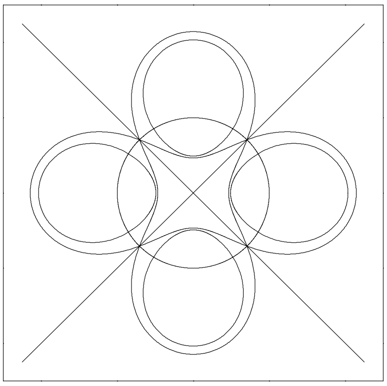

Figure 2. The plane of phase-adjusted spectral parameter $z\exp(-\mathrm{i}\gamma/2)$ and its typical subdivision into domains of analyticity of Jost functions at supercritical background levels of concomitant field $\mu\nu > 1$ (s.s. $\eta = +\pi/4$).

the equations for the boundary lines (93) are easily casted into the very transparent form

$$\exp(+2\chi + \eta)\cos 2\varphi = 0 \tag{97}$$

$$[\cosh 2\chi - \exp(\eta)\cos 2\varphi]\sinh 2\chi = 0 \tag{98}$$

$$\sinh 2\chi - \exp(\eta)\cos 2\varphi = 0 \tag{99}$$

$$\sinh 2\chi + \exp(\eta)\cos 2\varphi = 0 \tag{100}$$

$$[\cosh 2\chi + \exp(\eta)\cos 2\varphi]\sinh 2\chi = 0 \tag{101}$$

$$\exp(-2\chi + \eta)\cos 2\varphi = 0. \tag{102}$$

Here all parameters are understood to be real valued. Thus, relating to the plane of phase-adjusted complex spectral parameter $z\exp(-\mathrm{i}\gamma/2)$ we see that either of equations (97) or (102) determine two perpendicular straight lines intersected in the initial and infinitely far points and inclined with respect to the horizontal by the angles $\pm\pi/4$ (see figures 1 and 2 for illustrations). Equations (99) and (100) reproduce, respectively, the horizontal and vertical ovals (as in figure 1) or the horizontal and vertical dumbbells (as in figure 2). The factor $\sinh 2\chi$ in either of equations (98) or (101) is responsible for the unit circle (both in figures 1 and 2) whereas the factors embedded in square brackets are able to emerge as genuine lines only at positive η. These additional lines are shown in figure 2 as horizontally oriented and vertically oriented pairs of eggs and correspond to the square-bracketed terms of equations (98) and (101), respectively, taken at $\eta = \pi/4$. The discovered bifurcation in the arrangement of analyticity domains from one type (as in figure 1) to another type (as in figure 2) occurring at $\eta = 0$ indicates an essentially different structure of admissible solutions at $\eta < 0$ and $\eta > 0$ relating to our nonlinear model.

In the next section, we will study the influence of above bifurcation on the low-excitation (linear) spectrum when applied to some truncated version of our nonlinear model.

5. Truncated version of the nonlinear model and peculiarities of its low-excitation spectrum

As in the previous section, we will continue dealing with the reduction (81)–(84). Here, we must underline that its first two equations labelled by (81) wipe two field variables $K_{11}^{+}(n)$ and $K_{22}^{+}(n)$ out of the problem, fixing instead two originally unspecified gauge functions $b_{11}^{+}(n)$ and $a_{22}^{+}(n)$. The explicit expressions for $b_{11}^{+}(n)$ and $a_{22}^{+}(n)$ follows from equations (33) and (39), respectively, with the use of constraints (81) and some of formulae (15)–(28) where necessary. In general, the respective results occupy a great deal of space so we restrict ourselves only to their truncated version

$$b_{11}^{+}(n) = b_{11}^{+} - (a_{11}^{-} - a_{22}^{-}) f_{12}^{+}(n) g_{21}^{-}(n-1) \tag{103}$$

$$a_{22}^{+}(n) = a_{22}^{+} - (b_{22}^{-} - b_{11}^{-}) g_{21}^{+}(n) f_{12}^{-}(n-1) \tag{104}$$

survived after the coupling parameters a_{11}^{+} and b_{22}^{+} have been put to be zeros. Here, the quantities b_{11}^{+} and a_{22}^{+} are, in principle, set to be arbitrary functions of time, though in practice both of them can be safely removed from the model evolution equations by means of proper phase-adjusting gauge of field amplitudes $f_{12}^{+}(n)$, $g_{21}^{+}(n)$ and $f_{12}^{-}(n)$, $g_{21}^{-}(n)$.

Now introducing the notations

$$a_{11}^{-} - a_{22}^{-} = -\mathrm{i}\alpha \tag{105}$$

$$b_{22}^{-} - b_{11}^{-} = +\mathrm{i}\beta \tag{106}$$

and taking into account the general results (15)–(28) and (33)–(40) of section 2 we obtain the truncated version of our differential-difference nonlinear model in its explicit form:

$$\begin{aligned} &+\mathrm{i}\dot{\mu}(n) + \beta q_{+}(n) r_{+}(n) - \beta q_{-}(n) r_{-}(n) + \alpha q_{+}(n+1) r_{+}(n) - \alpha q_{-}(n) r_{-}(n-1) \\ &\qquad + \alpha q_{+}(n+1) r_{-}(n) \mu(n) - \alpha q_{+}(n) r_{-}(n-1) \mu(n) = 0 \end{aligned} \tag{107}$$

$$\begin{aligned} &+\mathrm{i}\dot{q}_{+}(n) + \alpha \mu(n) q_{+}(n) + \alpha q_{-}(n) - \alpha[\nu(n) - q_{+}(n) r_{-}(n)] q_{+}(n+1) \\ &\qquad + \beta[1 + q_{+}(n) r_{+}(n)] q_{-}(n-1) = 0 \end{aligned} \tag{108}$$

$$\begin{aligned} &+\mathrm{i}\dot{q}_{-}(n) + \beta \nu(n) q_{-}(n) + \beta q_{+}(n) - \beta[\mu(n) - q_{-}(n) r_{+}(n)] q_{-}(n-1) \\ &\qquad + \alpha[1 + q_{-}(n) r_{-}(n)] q_{+}(n+1) = 0 \end{aligned} \tag{109}$$

$$\begin{aligned} &-\mathrm{i}\dot{\nu}(n) + \alpha r_{+}(n) q_{+}(n) - \alpha r_{-}(n) q_{-}(n) + \beta r_{+}(n+1) q_{+}(n) - \beta r_{-}(n) q_{-}(n-1) \\ &\qquad + \beta r_{+}(n+1) q_{-}(n) \nu(n) - \beta r_{+}(n) q_{-}(n-1) \nu(n) = 0 \end{aligned} \tag{110}$$

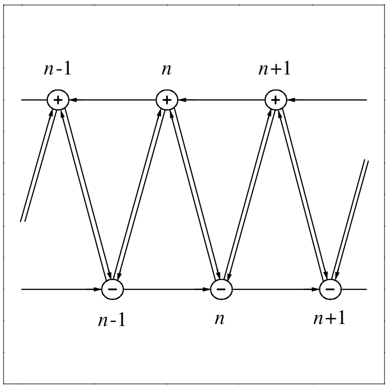

Figure 3. The three-cell fragment of a zigzag-runged ladder lattice associated with the truncated version of differential-difference nonlinear system. Every arrow pointing to a particular site indicates the linear or composite coupling between this site and the site where the arrow begins.

$$-\mathrm{i}\dot{r}_+(n) + \beta\nu(n)r_+(n) + \beta r_-(n) - \beta[\mu(n) - r_+(n)q_-(n)]r_+(n+1)$$
$$+\alpha[1 + r_+(n)q_+(n)]r_-(n-1) = 0 \qquad (111)$$

$$-\mathrm{i}\dot{r}_-(n) + \alpha\mu(n)r_-(n) + \alpha r_+(n) - \alpha[\nu(n) - r_-(n)q_+(n)]r_-(n-1)$$
$$+\beta[1 + r_-(n)q_-(n)]r_+(n+1) = 0. \qquad (112)$$

From these equations we see that the complex conjugation between $\mu(n)$ and $\nu(n)$ as well as between $q_+(n)$ and $r_+(n)$ and simultaneously between $q_-(n)$ and $r_-(n)$ can be ensured provided that α and β are taken as complex-conjugated $\alpha^* = \beta$.

Appealing to the amplitudes $q_+(n), r_+(n)$ and $q_-(n)$, $r_-(n)$ as to the transporting ones it is convenient to associate the model spatial geometry with a zigzag-runged ladder lattice, where indices + and − label two different straight legs of the ladder while n marks the unit cell. The rungs of such a ladder are arranged into a zigzag-like chain so that every lattice site is thought as intersection of two rungs and one leg (see figure 3 for illustration). Despite of their alleged similarity in the arrangements of lattice sites our present truncated model turns out to be substantially different from our earlier suggested model [6] when being truncated for comparison. The main distinction is the existence of the additional two-component dispersionless (in the linear sense) field $\mu(n)$ and $\nu(n)$ with a finite background. The intensity of this concomitant field regulate the strength of composite intersite coupling parameters $-\alpha\nu(n)$ and $-\beta\mu(n)$ along the chains for the Schrödinger field amplitudes $q_+(n)$, $r_+(n)$ and $q_-(n)$, $r_-(n)$. The spatial symmetry of such a type of longitudinal coupling is seen to be essentially broken so that the time evolution of $q_+(n)$ and $r_+(n)$ depends on the right-site amplitudes $q_+(n+1)$ and $r_+(n+1)$, respectively, whereas the time evolution of $q_-(n)$ and $r_-(n)$ depends on the left-site amplitudes $q_-(n-1)$ and $r_-(n-1)$, respectively. The linear couplings between the sites on opposite legs governed by the parameters α and β are more or less standard being analogous to the intersite resonant couplings known in the theory of molecular excitons [14, 15]. Due to their complexity, the parameters α and β are able to treat the external magnetic field perpendicular to the plane of ladder in terms of Peierls phase factors [16, 17].

It is interesting to note that the concept of composite intersite and self-site coupling parameters $-\alpha\nu(n)$, $-\beta\mu(n)$ and $+\alpha\mu(n)$, $+\beta\nu(n)$, respectively, allows one to describe the additional magnetic flux of essentially internal origin carried by the accompanying two-component field $\mu(n)$ and $\nu(n)$. The question whether or not this concomitant field has any reminiscence of Chern–Simons gauge fields [18, 19] could be examined by a separate comprehensive investigation. Below we will bring some arguments in favour of affirmative answer while considering the spectrum of low-amplitude excitations.

In so doing we will employ the complex-conjugated reduction (namely, $\alpha^* = \beta$, $\mu^*(n) = \nu(n)$, $q_+^*(n) = r_+(n)$, $q_-^*(n) = r_-(n)$) of truncated model (107)–(112) and assume additionally that parameters α and β are time independent. Then dropping the terms nonlinear with respect to Schrödinger amplitudes and replacing the concomitant field amplitudes by their background values (85) we observe that the equations for $\mu(n)$, $\nu(n)$ get trivialized while equations for $q_+(n)$, $r_+(n)$ and $q_-(n)$, $r_-(n)$ get linearized. As a consequence, the standard plane-wave substitutions

$$q_+(n) = C_+ \exp[\mathrm{i}(k - 2\vartheta)n - \mathrm{i}\Omega\tau] \tag{113}$$

$$q_-(n) = C_- \exp[\mathrm{i}(k - 2\vartheta)n - \mathrm{i}\Omega\tau] \tag{114}$$

with the phase 2ϑ specified through the parametrization

$$\alpha = \exp(\xi + \mathrm{i}\vartheta) = \beta^* \tag{115}$$

yield

$$\left[\Omega + 2\exp(\xi + \eta)\sin\left(\frac{k}{2}\right)\sin\left(\frac{k}{2} - \vartheta - \gamma\right)\right]^2 = 4\exp(2\xi)\cos^2\left(\frac{k}{2}\right)\left[1 - \exp(2\eta)\sin^2\left(\frac{k}{2} - \vartheta - \gamma\right)\right]. \tag{116}$$

Here, the parameters η and γ, though as previously being defined by expressions (94) and (95), are restricted to be time independent.

Equation (116) determines the stable linear spectrum only at negative η. In contrast at positive η the linear spectrum becomes unstable indicating a sophisticated spatial structurization of permissible solutions as compared with the familiar spatially uniform ansatz (113) and (114). These results of purely linear analysis are completely in lines with the bifurcation phenomena responsible for the crossover in the qualitative organization of analyticity domains of Jost functions.

The two-branch character of the linear spectrum (provided it exists) is stemmed from the number of structural elements in the unit cell, being precisely two elements for the case of the zigzag-runged ladder lattice.

Another significant implication of linear analysis follows from the additivity of phases ϑ and γ in the spectral equation (116) signalizing that the magnetic flux ϑ emerged from the external magnetic field and the magnetic flux γ carried by the concomitant field operate on an equal footing until the excitation level is sufficiently small and the background level of concomitant field is subcritical $\mu\nu < 1$.

6. Conclusion

Apart from its basic results briefly formulated in the abstract the present paper we believe gives a wide space for future investigations.

The main problem of course is the comprehensive development of inverse scattering technique (being rather nonstandard for the boundary conditions adopted here) with the aim to integrate the nonlinear model explicitly.

Another line of enquiry is to study the Lagrangian or the Hamiltonian structure of the model in terms of field amplitudes. To solve such a type of problems sometimes becomes an exceptionally nontrivial task. For example, the exact Hamiltonian formulation of multicomponent semidiscrete nonlinear Schrödinger systems [3, 5, 11, 17] appears to be still unknown [20, 21].

We are not sure whether the procedure developed here to produce the conservation laws covers all ramifications caused by the intricacy of auxiliary spectral problem and its possible variations due to boundary conditions. Thus, even in the general case of spectral operator (3) and evolution operator (9) with unspecified gauge functions $b_{11}^{+}(n)$ and $a_{22}^{+}(n)$ the model proves to corroborate early unnoticed but the very important conservation law

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\sum_{n=-\infty}^{\infty}\ln\left[1+\varkappa_{11}^{-}(n)\varkappa_{22}^{-}(n)-f_{12}^{-}(n)g_{21}^{-}(n)-f_{12}^{+}(n)g_{21}^{+}(n)\right]=0 \qquad (117)$$

akin to the law conserving the number of particles in semidiscrete nonlinear Schrödinger systems [1–3, 5, 6, 8, 11, 17]. This conjecture has been explicitly verified by the direct manipulations with the nonlinear equations taken at a_{11}^{+} and b_{22}^{+} being zeros. Another interesting observation concerning our model is empirically found on-site conservation laws. Indeed, the direct use of nonlinear system under the previously mentioned assumptions shows that every particular term under the summation symbol in either of formulae (76) and (78) is conserved. Anyway, the problem of conservation laws does not seem to be entirely closed.

Finally, one may expect a number of interesting reductions connected with a variety of possible boundary conditions giving rise to substantial modifications of auxiliary spectral problem and perhaps to unexpected metamorphoses of the inverse scattering technique.

In this context, it would be interesting to consider one more aspect of original discrete spectral problem concerning its plausible continuum limit taken properly to preserve the integrability of spatially continuous counterpart of nonlinear differential-difference system. Maybe such an analysis could provide us with the clear physical interpretation of involved field amplitudes and the nonlinear integrable system as a whole. We are unable to answer this question *a priori* inasmuch as the strategy of present work has not been based on a discretization of any prototypic continuous integrable model with already known physical properties. Speaking plainly, finding out the self-consistent continuous version of our model is still a matter for separate research.

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